The Historical Development of the Fundamental Theorem of Calculus And Its Implication in Teaching

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Introduction

Differing from conventional instructional order, historical development of integral calculus preceded that of differential. Origins of integral calculus date back to as early as ancient Greece, in efforts to find area, volume, and arc length. Basic idea of integration is considering an area as approximated by the sum of areas of many thin parallel rectangular strips—as the number of strips increases infinitely, width of each strip approaches zero. One can then calculate area as the limit approached by the sum of the areas of these strips. On the other hand, differentiation was initiated from the problem of deriving slope of the tangent to a curve and calculating instant velocity. Key point for both concepts is treating instant rate of change as the ultimate value of average rate of change, not explicitly recognized until the 17th century. Intuitively, derivative and definite integral are two seemingly disparate notions: one based on the limit of a sum of a growing number of vanishing elements, another on the limit of a difference quotient. The Fundamental Theorem of Calculus manifests the fantastic mutually inverse relationship between the two, in the same sense of addition and subtraction, or multiplication and division. Discovery of the striking inverse relationship between these concepts is deemed the root idea sustaining the whole of calculus, and it should be noted that over a century of investigation was needed to attain its present status. The significance of establishing the link is pinpointed by Howard Eves: "In any collection of GREAT MOMENTS IN MATHEMATICS, the discovery of the fundamental theorem of calculus would surely appear" (Eves, 1983, p.38). The chief foci of this paper are sketching its brief history and discussing classroom activities introducing this great moment of mathematics to Taiwanese college students.

The Development of the Fundamental Theorem of Calculus

Newton and Leibniz share the honor of invention of calculus and independently proposed the Fundamental Theorem of Calculus, yet they were not the first cognizant of the inverse relation between processes of integration and differentiation. From a current point of view, several earlier mathematicians, either implicitly or explicitly, had captured the inverse essence of these concepts. Some particular cases even had been established. Newton's famous motto holds: "If I have seen further than others, it is because I had stood upon the shoulders of giants." It is the time to find out what these giants are.

The time before Isaac Barrow

In the early 17th century, Evangelista Torricelli recognized the inverse relation between integration and differentiation holding for generalized parabolas. In modern terms, Torricelli actually showed that

$$\frac{d}{dx}\int_0^x x^n dx = \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$$, where n is a natural number.

In 1655, John Wallis considered the more general cases of n as a rational number and negative exponents in his *Arithmetica Infinitorum* (*The Arithmetic of Infinites*), believed to have exerted decisive influence on Newton's early mathematical development. In fact, Fermat and Torricelli earlier established Wallis's work on the case of rational number, yet their works were never published until somewhat later (Mahoney, 1973). Initially, Fermat viewed constructing a straight-line segment equal in length to a given algebraic curve as impossible. Shortly before 1660, as infinitesimal techniques were increasingly applied, this belief turned questionable (Boyer, 1959). First rectification of a curve was that of semi-cubical parabola $y^2 = x^3$ in 1657, proposed by William Neil (Appendix). Upon hearing of Neil's work, Fermat was motivated to carry out rectification of the more general semi-cubical parabola $my^2 = x^3$. As seen in Figure 1, for any point P on the curve $my^2 = x^3$ with abscissa OQ (length as a) and ordinate PQ (length as b), Fermat showed how the length of subtangent RQ (length as c) is $2a/3$. Let ordinate $P'Q'$ to the tangent line be erected at distance e from PQ, the length of segment PP', in terms of a and e, is $PP' = e\sqrt{\frac{9a}{4m}+1}$. Note that, for sufficiently small values of E, point P' can be seen as on the curve, whose length, in this manner, may be treated as the sum of segments like P P'. Meanwhile, by virtue of the fact that $PP' = e\sqrt{\frac{9a}{4m}+1}$, total sum of these segments actually is the area under the parabola $y^2 = \frac{9x}{4m}+1$. It is therefore obvious that the quadrature can be obtained as long as the length of the curve is determined.

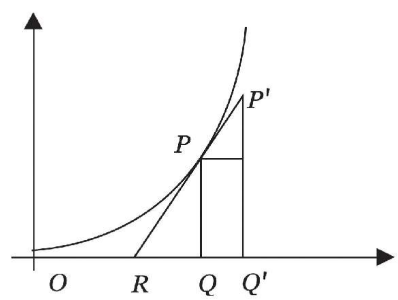


Figure 1

Apparently, Fermat reduced a problem of rectification by connecting tangents and the question of quadratures. Surprisingly, for all his deft use of infinitesimals in a variety of areas, he still failed to recognize this critical relation, denying himself the honored title of "true inventor of the calculus" (Boyer, 1959). The man first overtly aware of generality of the Fundamental Theorem of Calculus was James Gregory in 1668, exerting a significant influence on Isaac Barrow's work.

The Work of Isaac Barrow

Following Galilei's pioneering work, study of the time-motion curve probably led Barrow to intuitive understanding of the inverse relation between tangent and quadrature problems. In *Lectiones opticae et geometricae* (*Geometrical Lectures*) of 1669, Barrow proposed the earliest and clearest, though incomplete, version of the Fundamental Theorem of Calculus. His result may be described as follows (Edwards, 1979; Eves, 1983):

> Let the y- and z-axis be oppositely oriented as shown in Figure 3. Given an increasing positive function $y = f(x)$, denote by $z = A(x)$ the area between the curve $y = f(x)$ and the segment $[0, x]$ along the x-axis. Given a point $D(x_0,0)$ on the x-axis, and let T be the point on the x-axis such that $DT = DF/DE = A(x_0)/f(x_0)$. Then the line TF touches the curve $z = A(x)$ only at the point $F(x_0, A(x_0))$.

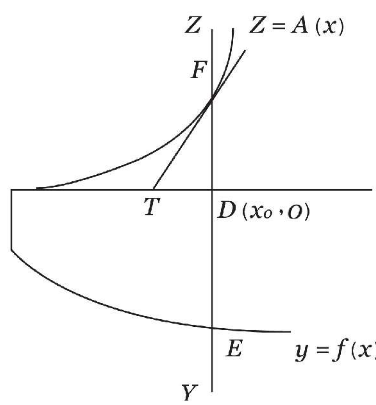


Figure 2

Barrow concluded the theorem merely by asserting that line TF touches curve $z = A(x)$ only at the point $F(x_0, A(x_0))$ rather than explicitly indicating TF as the tangent to $z = A(x)$. Since slope of TF is $\dfrac{DF}{DT} = \dfrac{A(x_0)}{A(x_0)/f(x_0)} = f(x_0)$, if Barrow further asserted TF is a tangent line to curve $z = A(x)$; this result would lead to a conclusion that $A'(x_0) = f(x_0)$, the Fundamental Theorem of Calculus. Barrow typically dealt with tangent-quadrature problems in a geometrical fashion; this cumbersome geometrical approach may have precluded his gaining insight into this theorem.

Newton and Leibniz's work on the Fundamental Theorem of Calculus

Contrary to previous infinitesimal techniques mostly based on the determination of area as a limit of a sum, Newton considered the rate of change of a desired area and calculated said area via anti-differentiation (Edwards, 1979; Struik, 1969). Let $A(x)$ denote the area BCD under curve $y = f(x)$ (Figure 3) and regard this area as vertically swept out by segment BC moving to the right with unit velocity, i.e., $\dot{x} = 1$.

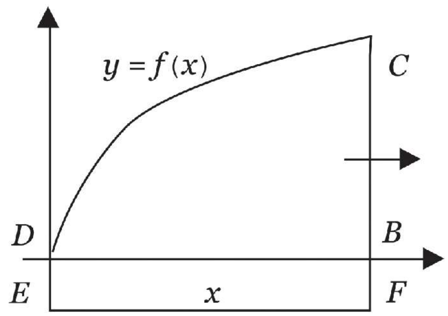


Figure 3

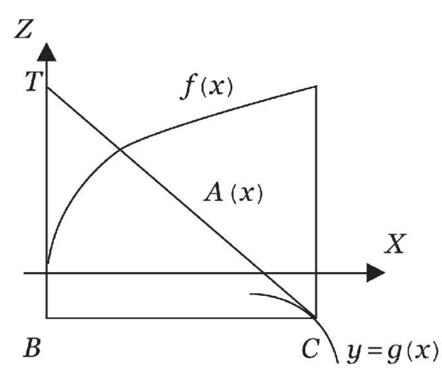
Extend CB to F, so that $BF = 1$, and complete the rectangle $BDEF$. Newton then asserted fluxions of areas BCD and $BDEF$ should be BC and BF, respectively ($\dot{y} = BC$, $\dot{x} = BF$). Thus, the derivative of the area under the curve $y = f(x)$ is $y = f(x)$ itself:

$$\frac{\dot{y}}{\dot{x}} = \dot{y} = f(x).$$

Obviously, Newton's approach is dynamic in nature. Nevertheless, despite his crucial insight into this important relation, Newton did not yield rigorous proof.

On the other hand, as a logician and philosopher, Leibniz delayed formal study of mathematics until 1672, when he was sent to Paris on a diplomatic mission. Similar to Fermat's fashion, Leibniz studied rectification problem by means of a problem of quadrature. In a 1677 manuscript, Leibniz introduced the Fundamental Theorem of Calculus:

Figure 4



Given curve $z = f(x)$ (Figure 4), if it is possible to find the curve $y = g(x)$ such that the slope of tangent $\frac{TB}{BC} = \frac{z}{k}$, where k is a constant, then $\frac{dy}{dx} = \frac{TB}{BC} = \frac{z}{k} \Rightarrow zdx = kdy$, so the area under the original curve is $\int zdx = k\int dy = ky$. A quadrature problem was thus reduced to inverse tangent problems. Namely, in order to find the area under the curve with ordinate z, it suffices to find a curve whose tangent satisfies condition $\frac{dy}{dx} = z$. Setting $k = 1$ and subtracting the area over [0, a] from that over [0, b], we then obtain

$$\int_a^b z dx = y(b) - y(a).$$

In addition to borrowing from Fermat's approach, Leibniz's idea here is also quite akin to Neil's use of auxiliary curve while solving rectification problems (Appendix).

A Conjecture about Archimedes' Work on the Fundamental Theorem of Calculus

It is widely held that the notion of the Fundamental Theorem of Calculus was first acknowledged in the 17th century. Nonetheless, it could date back to ancient Greece if we look at Archimedes' work in more detail (Eisenberg & Sullivan, 2002; Grattan-Guinness, 1997). In his *Measurement of A Circle*, Archimedes derives area A of a circle by saying:

> *The area of any circle is equal to a right-angle triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle.*

What Archimedes said here is that the following two figures have the same area.

Circle of radius r and circumference C — Right triangle of base r and height C

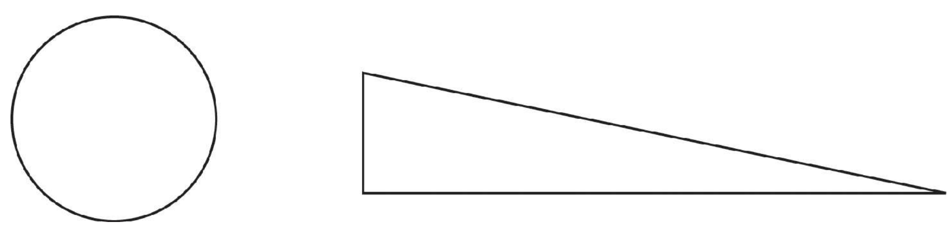


Figure 5

Once the result is obtained, we see the area A of a circle with radius r equals $\frac{1}{2}(rC) = \frac{1}{2}r(2\pi r) = \pi r^2$. How did Archimedes get this idea? What thought underlies this proposition? Contrary to his reductio ad absurdum employed to prove this theorem, Archimedes perhaps viewed a circle as a combination of infinitely many concentric circles, so that its area can be regarded as an infinite sum of the "width" of circumferences. He then got all circumferences straight and piled them up to form a right triangle whose height is r and base is the longest circumference C (Figure 6), as Abraham bar Hiyya ha-Nasi interpreted (Grattan-Guinness, 1997).

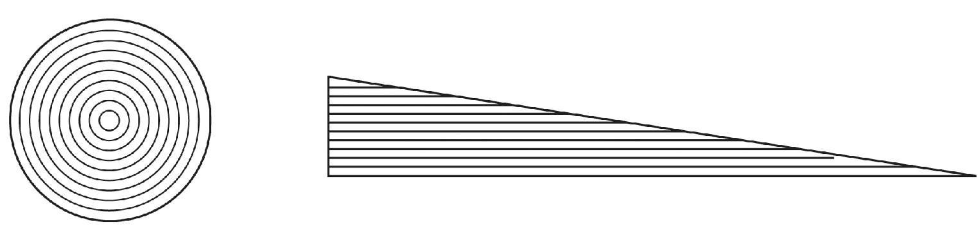


Figure 6

If this conjecture stands, Archimedes would become the first to recognize the mutually inverse relation between integration and differentiation. While viewing a circleas a combination of infinitely many concentric circles, Archimedes regarded circumference $C(r)$ as instant rate of change of area $A(r)$ of a circle—i.e., $dA(r)/dr = C(r)$. Conversely, when evaluating area of the right-angle triangle with height r and base C, he actually summed up infinitely many of $C(r)$ to get the area of a circle with radius r—i.e., $\int_0^r C(r)dr = A(r)$, essentially the Fundamental Theorem of Calculus. If this is the case, it is curious why Archimedes went no further. Did he fail to perceive this vital fact? Was he unaware of the importance of the notion of rate of change at this time? It may be also interesting to know whether any 17th century mathematicians got insight from Archimedes' work.

Emergence of the Fundamental Theorem of Calculus affords us a clear-cut example between *discovery* and *recognition* of significance (Eves, 1983). Of all mathematicians prior to Newton and Leibniz, Fermat and Barrow exhibited the closest thinking to this newborn discipline. Fermat appeared to realize the inverse relation between these types of problems, but seemingly restricted his attention to solving geometrical problems. Barrow provided a geometric theorem elucidating the inverse relationship, yet failed to recognize the key essence of his result. He reduced inverse-tangent problem to quadratures, yet did not go reverse direction. Besides, in some sense, his geometrical approach indicated a retreat to the idea of indivisibility of Cavalieri (Boyer, 1959). Newton and Leibniz's contribution not only conceptualized the Fundamental Theorem of Calculus as a crucial fact, but also effectively employed it to advance earlier infinitesimal techniques to a powerful algorithmic instrument for systematic calculation. It also should be noted that rigorous structure was lacking both in Newton and Leibniz's proofs of the Fundamental Theorem of Calculus, in that the knowledge for the foundation of calculus was not well established during their era, something for which they were not responsible. Rigorous proof was not available until Cauchy, more than 100 years later.

The Implication In Mathematics Teaching

Discovery of the Fundamental Theorem of Calculus cannot be seen merely as some effective methods created for solution of problems involving tangents and quadratures. Its half-century-long evolution not only shows a typical mode of forming of mathematical knowledge but also reflects human facets in constructing this great scientific endeavor. In an epistemological point of view, this historical event is quite worthy of being taught in school. In my historical approach college calculus course, instead of presenting students the statement and proof of the Fundamental Theorem of Calculus directly, I introduced the aforementioned historical processes to my class and investigated how students reacted to it. First of all, I assigned handouts regarding the development of the Fundamental Theorem of Calculus, including Fermat, Barrow, Newton, and Leibniz's approaches shown above. All students were asked to study and think about the handouts. One week later, several students were chosen at random to explain the historical methods on the board and were also reminded that the formal proof did not appear until Cauchy. This classroom activity was designed to offer students an opportunity to realize various fashions and approaches of thinking among mathematicians. Following the presentations, I reviewed Archimedes' method of deriving area of a circle (students were aware of it at the outset of the course) and proposed the aforementioned plausible conjecture. It was hoped, in this manner, they would attain holistic grasp of the significance and historical context of the Fundamental Theorem of Calculus. The context categorized the role of each mathematician as:

(1) Archimedes: potentially recognizing the theorem;
(2) Fermat: connecting the concepts of tangents and the question of quadratures;
(3) Barrow: only one step short to the discovery of the theorem;
(4) Newton: discovering and interpreting the theorem by using dynamic approaches;
(5) Leibniz: extending Neil and Fermat's methods to attain the result;
(6) Cauchy: proving the theorem.

Thereafter, students were asked to write down (a) their realizations and thoughts about the discovery and progress of the Fundamental Theorem of Calculus and (b) who made the most significant contribution. Major responses to the discovery of the Fundamental Theorem of Calculus are listed below.

Mathematical knowledge is attributed to mathematicians' constant effort.

Most significant perspective among students was the progress of mathematical concepts is a continuing effort made by mathematicians. As one of students Li indicated:

> "The progress of calculus was quite complicated and sophisticated. Fermat's method of tangent and Archimedes' use of concentric circles were amazing! But one thing for sure is the ultimate result was generated by means of mathematicians' enduring efforts."

Another student Chen professed:

> "When I see so may mathematicians' approaches, I cannot but marvel that the Fundamental Theorem of Calculus we learn today was obtained through so many mathematicians' hands…No wonder Newton said: 'If I have seen farther, it is by standing on the shoulders of giants'!"

Further, many students regarded the growth of mathematical knowledge as a relay race, as manifested in Shao's description:

> "Calculus must be developed from geometry, I guess. Beginning with Archimedes' approach of deriving the area of a circle by means of the area of a triangle, then Torricelli, Fermat (connecting tangent slope and area), until Barrow, Newton, and Leibniz's insight into the problem, calculus seemingly experienced multifarious features and manners. The process was a little bumpy. However, after a series of relay races, proof of the Fundamental of Calculus was finally given."

Aforementioned statements suggest students may achieve an appropriate understanding about the role human beings play in the making of mathematics.

Formation of mathematical knowledge is a long-term accumulation process.

Besides recognizing mathematicians' role, some students even understood that ongoing incubation is unavoidable for the growth of mathematical knowledge, as seen in Gerng's response:

> "I can only say it (the progress of mathematics) is a snaky way. Dating from Archimedes' likely discovery to Cauchy's proof, it took thousands of years. Some were one step away from the discovery, and some proposed result without giving proof…These concepts might not be immediately understood by mathematicians at the time. Therefore, to some degree, long-term development is necessary."

Student Lin displayed a sophisticated view that development of mathematics is an endless course:

> "The developmental process of calculus is like a puzzle, accumulated slowly. Starting from the area deriving by using infinity and limit, then discovering the relationship between differentiation and integration, until a rigorous proof emerged. Almost all mathematical methods reveal that *the progress of mathematics is a procedure of the heir of ancient sages and initiation of posterity, a never-ending puzzle* [italics added]."

The phrase "a never-ending puzzle" to a great extent appropriately depicts the dynamic image of the formation of mathematical knowledge.

Discovery may not necessarily lead to verification.

Students' responses also showed they might have realized the establishment of mathematical facts may not be trouble-free at mathematicians' hands and recognizing a concept guarantees nothing about its validity. As Huang indicated:

> "Throughout the whole development of the Fundamental Theorem of Calculus, it can be seen that some had unconsciously found the law; some studied again the same topic, motivated by others' fresh insight. Even the discoverer may fail to prove the result."

A student, Wu, cited failure to give proof as attributable to personal blind spots, while Chiou proposed a probable answer to this issue:

> "Several mathematicians could nearly become the creator of calculus. Unfortunately, some important details were missing. I guess *they probably focused on some other problems then and merely treated it as a problem-solving tool* [italics added]. Thus a concrete organized study was lacking."

We are too often eager to probe facts hidden behind appearances. Chiou's response alerts us that historical study should not impose our own stories on the evidence of the past.

Newton and Leibniz made the most significant contributions.

While students' opinions on the mathematician with the most crucial contribution to the Fundamental Theorem of Calculus were varied, a clear image emerged (12 of the 36 students) crediting Newton and Leibniz, as seen in Liao's claim:

> "Newton and Leibniz made the most noteworthy contribution, since they identified the mutually inverse relationship between differentiation and integration, which was hard to discover. Barrow and Archimedes almost found it but failed to explicitly propose it…Caucy gave the proof, but he couldn't make it without Newton's work."

Lin praised Newton and Leibniz's achievement by saying:

> "The situation at that time could be described as all is ready but a timely kick. It was they who made that critical shot, revealing all mysteries. But nobody identified this point before them. Hence I think they made the major contribution."

Moreover, a large portion of students (10 of the 36) considered the accomplishment should be accorded to Fermat. Chen claimed that,

> "In many aspects, Fermat's approaches were so similar to modern ones. Though he did not point out explicitly, a rudiment was formed, paving the way for Newton and Leibniz. Calculus couldn't be discovered so early without him."

Students' replies highlight a view that enlightened ideas are the most valuable in generating mathematical knowledge. Verification of knowledge, which professional mathematicians frequently stress, was seemingly almost ignored.

Conclusion

Stressing humanistic value in the making of mathematics is a central theme of this article. As taught in traditional curriculum, emphasizing literacy of computational skills and its utility in the real world, mathematics has long lost its human face. Reviewing history of mathematics would yield a clear image of mathematical knowledge as motivated either by environmental stress (sociocultural factors) or hereditary stress (mathematicians' intellectual curiosity across generations). Both stresses coincidentally indicate the indispensable role of humans at different places and in various times. Besides imparting skills, education is an important means for transmitting human culture and values across generations, which have often been less focused, even totally skipped, in our educational systems. For eliciting students' interest of learning mathematics, teachers tend to emphasize the utility of mathematics, yet hide from their students the excitement and intrinsic spirit of the discipline as a result. A humanistic approach thus has been suggested to remedy this sad situation (Tymoczko, 1993 Davis, 1993).

Classroom activities introduced in this article convey a belief that mathematics is a discipline with a human perspective and history, putting it among the humanities. Thus to introduce students to humanistic mathematics is to show them human intellectual adventure in mathematics, challenging dogmatic teaching styles requiring them to follow lecture and practice recipe-like drill (Hersh, 1993). Mathematics is the creation of concepts and exploration in facts. Bronowski (1965) indicated: "Science is not a mechanism but a human progress, and not a set of findings but a search for them" (p. 63). It should be stressed that the search for the beautiful result of the Fundamental Theorem of Calculus not only is the great moments of mathematics but that of humanity.

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Appendix

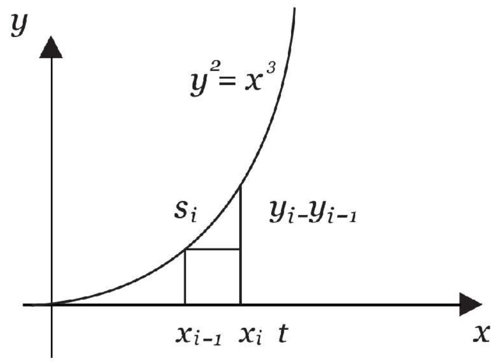

Figure 7

To calculate the length of curve $y = f(x)$ over $[0, t]$ (Figure 7), Neil subdivided the interval $[0, t]$ into an indefinitely large number n of infinitesimal subintervals, the *ith* one being $[x_{i-1},x_i]$. Let s_i denote length of the i_{th} piece of the curve $y = f(x)$ joining the corresponding points (x_{i-1}, y_{i-1}) and (x_i, y_i) then $s_i \cong [(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2]^{1/2}$. The length of the curve is therefore given by $s \cong \sum_{i=1}^{n} [(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2]^{1/2}$. Let A_i denote the area under the curve $y = f(x)$ on $[0, x_i]$, for computing the value of s, Neil introduced a curve $z = g(x)$ such that the area A_i over $[0, x_i]$ is

$$A_i = \int_0^{x_i} g(x)dx = f(x_i) = y_i$$

It follows that $y_i - y_{i-1} = A_i - A_{i-1} \cong g(x_i)(x_i - x_{i-1})$. We therefore have

$$s \cong \sum_{i=1}^{n} [1 + (g(x))^2]^{1/2} (x_i - x_{i-1})$$

$$\Rightarrow s = \int_0^a \sqrt{1 + [g(x)]^2}\, dx.$$

We can see that the proper choice of the auxiliary curve is $g(x) = f'(x)$ and the link between quadratures and tangents indeed was implicitly shown by Neil's method.